# Choosing Your Nonmonotonic Logic: A Shopper's Guide

ULF HLOBIL[1]

**Abstract:** The paper presents an exhaustive menu of nonmonotonic logics. The options are individuated in terms of the principles they reject. I locate, e.g., cumulative logics and relevance logics on this menu. I highlight some frequently neglected options, and I argue that these neglected options are particularly attractive for inferentialists.

**Keywords:** Nonmonotonic Logic, Inferentialism, Cumulative Transitivity, Cautious Monotonicity, Relevance Logic

Nonmonotonic logics are logics in which Weakening or Monotonicity (MO) sometimes fails.

MO If $\Gamma \vdash A$, then $\Gamma, \Delta \vdash A$.

Given that you are reading this, you are presumably in the market for a nonmonotonic logic. In this paper, I will offer some guidance by giving you a complete menu of options. The different options are individuated by the principles they reject. I will highlight some frequently neglected options that are attractive for inferentialists.

I use the snake-turnstile "$\mid\!\sim$" to talk about nonmonotonic consequence. I take permutation and contraction for granted by working with sets on the left (and the right, in multiple conclusion logics) of the turnstile.

## 1 What do you need?

In order to know which nonmonotonic logic is right for you, we must know what you want your nonmonotonic logic *for*. There are two families of reasons for wanting a nonmonotonic logic. Either you want to get more

[1]I would like to thank Robert Brandom, Daniel Kaplan, Shuhei Shimamura, Rea Golan, and everyone who supported the Pittsburgh research group on nonmonotonic logic over the years.

conclusions than classical logic gives you, or you want to get fewer (or both). The logics that are typically called "nonmonotonic logics" give us more conclusions than classical logic.[2] They add risky inferences, like the following:

(a) Zazzles is a cat. So Zazzles has four legs.

(b) I let go of this object $h$ meters above ground. So it will hit the ground with $\sqrt{2gh}\ \frac{mtr}{sec}$.

These inferences are obviously not classically valid; and they are defeasible. If we add to the first one the premise that Zazzles has (tragically) lost a leg in an accident, the conclusion no longer follows. And if we add to the second inference the premise that the object is a bird, the conclusion no longer follows. Nevertheless, the conclusions intuitively follow from the premises. Nonmonotonic logics in the first family try to capture this intuition.

Let us now turn to the second kind of motivation. Relevance logicians[3] reject MO in order to avoid fallacies of relevance, like the result that "It is not the case that if you are a philosopher, you are dumb" entails "If you are dumb, the moon is made of cheese." To avoid such results, relevance logicians restrict the classical principle that a set entails its elements (containment or CO), while accepting that every sentence entails itself (reflexivity or RE).

CO If $A \in \Gamma$, then $\Gamma \mid\!\sim A$.

RE $A \mid\!\sim A$.

In the atomic case, e.g., the set $\{p, q\}$ doesn't entail $p$. For relevance logicians, this is justified because $q$ is not relevant to $p$, where the notion of relevance is usually spelled out in terms of variable sharing.

Given these two families of motivations, any shopping tour should start with the question: Do you care about allowing risky inferences? Or do you want to avoid fallacies of relevance?

If you merely want to avoid fallacies of relevance and don't care about risky inference, I don't have any new insights to offer. You should shop around to find a relevance logic you like (see Anderson & Belnap, 1975; Dunn & Restall, 2002). If you care about both, relevance and risky inferences, I may have something new for you (see NM-LR below).

[2] Sometimes some other monotonic logic is used as the lower limit.

[3] For my present purposes, linear logic and similar logics count as relevance logics.

If you want to codify risky inferences and don't care about relevance, there are many logics on the market that you might like, e.g., preferential logics (Kraus, Lehmann, & Magidor, 1990), default logics (Reiter, 1980), adaptive logics (Batens, 2007), argument-based approaches (Dung, 1995), etc. In fact, there is a confusing variety of such logics. They are often categorized according to the technical machinery they use (Strasser & Antonelli, 2016). Preferential logics, e.g., use partial orders over worlds. Default logics use default rules. Adaptive logics use sets of abnormalities. Argument-based approaches use graphs in which the nodes are arguments. This is often not very helpful because you may not care what technical machinery is running under the hood of your logic.

I want to look at nonmonotonic logics from a more abstract perspective. This will allow us to explore the logical space in which nonmonotonic logics are located in terms of philosophical views about logic and principles you may want your logic to satisfy, where this includes structural principles and principles governing connectives.

## 2 The full menu

You cannot get a nonmonotonic logic without having to give up some principles that many find desirable. The good news is that you get a choice regarding which principles you want to give up. In this section, I will go through some of these choices. The result will be an exhaustive (but not exclusive) classification of nonmonotonic logics into seventeen types.

We can think of these seventeen types as generated by four choices. At each choice point, you must reject at least one of a given set of principles. The first choice (labeled [C1] in Figure 1) is that between rejecting CO and rejecting Mixed-Cut.

Mixed-Cut If $\Gamma \mid\!\sim A$ and $\Delta, A \mid\!\sim B$, then $\Gamma, \Delta \mid\!\sim B$.

You must choose between these two because CO together with Mixed-Cut implies MO. For, since by CO $B, A \mid\!\sim B$, if we have $\Gamma \mid\!\sim A$, we get $\Gamma, B \mid\!\sim A$ by Mixed-Cut. If we reject CO, we are in the area of relevance logics (in a very broad sense). If we reject Mixed-Cut, we reach our next choice point.

The second choice (labeled [C2] in Figure 1) is that between rejecting CO or the Deduction-Detachment Theorem (DDT) or Cumulative Transitivity (CT, aka Cut).

DDT $\Gamma \mid\!\sim A \to B$ iff $\Gamma, A \mid\!\sim B$.

CT If $\Gamma \mid\!\sim A$ and $\Gamma, A \mid\!\sim B$, then $\Gamma \mid\!\sim B$.

We cannot have all three because, by CO, $\Gamma, A, B \mid\!\sim A$ and so, by DDT, $\Gamma, A \mid\!\sim B \to A$. This means that we can get $\Gamma, B \mid\!\sim A$ from $\Gamma \mid\!\sim A$ by CT and (left to right) DDT (see Arieli & Avron, 2000).

The third choice (labeled [C3] in Figure 1) is one between rejecting the principle that everything implies all instances of the law of excluded middle (PEM), Cautious Monotonicity (CM), and a principle that I shall call Premise Fission (PF), which is basically reasoning by cases (as an invertible metarule).

PEM $\Gamma \mid\!\sim A \vee \neg A$.

CM If $\Gamma \mid\!\sim A$ and $\Gamma \mid\!\sim B$, then $\Gamma, B \mid\!\sim A$.

PF $\Gamma, A \vee B \mid\!\sim C$ iff $\Gamma, A \mid\!\sim C$ and $\Gamma, B \mid\!\sim C$.

To see that we must reject one of these, suppose again that $\Gamma \mid\!\sim A$. By PEM, $\Gamma \mid\!\sim B \vee \neg B$. By CM, $\Gamma, B \vee \neg B \mid\!\sim A$. By PF, $\Gamma, B \mid\!\sim A$.[4]

The fourth choice (labeled [C4] in Figure 1) concerns conjunction and disjunction. We must reject either PF, or Distribution (DI), or what I call Premise Fusion (FU).

DI If $\Gamma, A \vee (B \& C) \mid\!\sim D$, then $\Gamma, (A \vee B) \& (A \vee C) \mid\!\sim D$.

FU $\Gamma, A \& B \mid\!\sim C$ iff $\Gamma, A, B \mid\!\sim C$.

PF requires an additive disjunction left-rule. FU requires a multiplicative conjunction left-rule. An additive disjunction doesn't distribute over a multiplicative conjunction (on the left of the turnstile).[5] To see this, notice that if $\Gamma, A \mid\!\sim D$ and $\Gamma, B, C \mid\!\sim D$, we get $\Gamma, A \vee (B \& C) \mid\!\sim D$ by FU and PF. But if also $\Gamma, B, A \not\mid\!\sim D$, we cannot get $\Gamma, (A \vee B) \& (A \vee C) \mid\!\sim D$.

We can use these four choice points to give an exhaustive categorization of nonmonotonic logics. If we suppress some of our choices for relevance logics (by leaving out the CM, PEM, PF choice), we can distinguish seventeen types of nonmonotonic logics. They correspond to the leaves of the tree in Figure 1.

[4] This argument was brought to my attention by Dan Kaplan. It is used in the literature, e.g., by Arieli and Avron (2000).

[5] More precisely, an additive disjunction distributes over a multiplicative conjunction across the turnstile. In a multiple conclusion setting, we also get: If $\Gamma \mid\!\sim \Delta, A \vee (B \& C)$, then $\Gamma \mid\!\sim \Delta, (A \vee B) \& (A \vee C)$. However, we don't get DI.

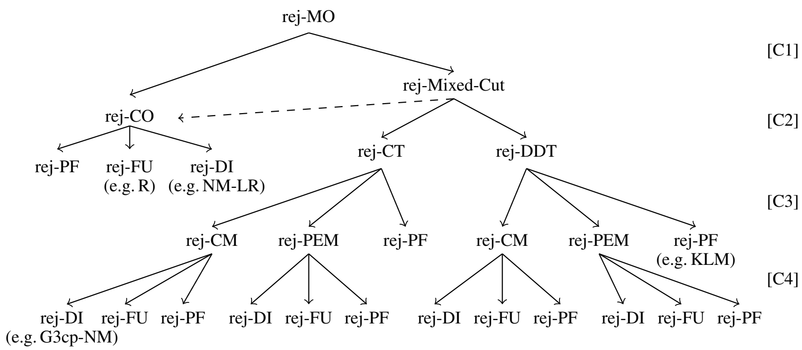


Figure 1: Types of nonmonotonic logic

Figure 1 should be read as follows: Every nonmonotonic logic must reject all the principles that occur on at least one complete branch of the tree. We can label the seventeen types by the principles that they reject. The relevance logic R, e.g., belongs to the type *Rej(CO,FU)*. For CO doesn't hold in R and the conjunction of R doesn't obey FU (even though the fusion operator does).[6] The dashed arrow indicates that the choice between rejecting CT and DDT is only forced if you accept CO. In fact, logics like NM-LR below obey CT and DDT; this is possible because CO doesn't hold in NM-LR.

Of course, a logic can always reject more principles than what the tree in Figure 1 requires. Hence, a logic can belong to several of our seventeen types. The logic NM-LR below, e.g., belongs to *Rej(CO,DI)* and *Rej(CO,FU)* because it rejects DI and FU. These types are, however, exhaustive in the sense that every nonmonotonic logic must belong to at least one of our seventeen types.

Let me illustrate the use of this categorization. Cumulative logics, in the sense of KLM (Kraus et al., 1990), accept CO and CT. This puts cumulative logics in the branch that rejects DDT. Cumulative logics also accept CM, and they are supra-classical, which means that they must accept PEM. So, cumulative logics belong to the type: *Rej(Mixed-Cut,DDT,PF)*. That tells us that Premise Fission cannot hold in cumulative logics.

[6]Once we have different kinds of conjunction (and perhaps disjunction) around, there is a question what principles like PF, FU, and DI mean. The answer is that what matters is just that PEM, PF and DI use the same disjunction and FU and DI use the same conjunction.

In preferential logics, which are a kind of cumulative logic, PF fails because we get $\Gamma, A \vee B {\mid\!\sim} C$ in cases in which $\Gamma, A {\mid\!\not\sim} C$. As an example, suppose that, in our favorite preferential logic, "Zazzles is a cat" nonmonotonically implies "Zazzles has four legs." By supra-classicality "Zazzles is a cat" implies "Zazzles either did or didn't have an accident in which he lost a leg." By CM, the set {"Zazzles is a cat", "Zazzles either did or didn't have an accident in which he lost a leg"} implies "Zazzles has four legs." That seems counter-intuitive. After all, "Zazzles is a cat. And Zazzles had an accident in which he lost a leg" clearly doesn't imply "Zazzles has four legs." As is clear from Figure 1, cumulative logics cannot avoid such cases.

## 3 Motivating choices

The tree in Figure 1 gives us a complete menu of options. Some of these options have well-known representatives, like KLM or the logic R. Below I will focus on options that have not been explored in the literature. I will present one logic of type *Rej(Mixed-Cut,CT,CM,DI)*, which I call G3cp-NM, and one logic of type *Rej(CO,DI)*, which I call NM-LR. The latter logic stands to LR as G3cp-NM stands to classical logic. Before we get into any technical details, however, we should reflect on our menu of options from a philosophical perspective.

The choices that KLM makes, i.e. *Rej(Mixed-Cut,DDT,PF)*, can be seen as embodying two priorities:[7] (a) rejecting as few structural principles as possible, and (b) staying supra-classical. To reject CM or CT would be to reject a structural principle where you could instead reject principles about particular connectives, namely DDT and PF respectively. So this would go against priority (a). Rejecting PEM instead of PF would preclude KLM from being supra-classical and, hence, go against priority (b).

Logics of the type *Rej(Mixed-Cut,CT,CM,DI)*, like the logic NM-G3cp below, can be seen as embodying the idea that we shouldn't reject principles regarding the behavior of connectives if we can instead reject structural principles. One way to flesh out this view is to say that satisfying DDT is part of what it means for something to be a conditional. Similarly, satisfying PF is part of what it means for something to be a disjunction. If we also want to stay supra-classical and, hence, accept CO and PEM, then it is al-

[7]Of course, I don't want to do the KLM advocate's introspection for her. And there can be other priorities that might motivate the choices KLM makes. But (a) and (b) are at least a natural interpretation.

ready settled that we must reject CT and CM. If we now add that satisfying FU is part of what it means to be a conjunction, we arrive at a logic of type *Rej(Mixed-Cut,CT,CM,DI)*.

This comparison brings out that navigating our decision-tree can be seen as an exercise in trading off structural principles that one may take to be constitutive of consequence against principles that one may consider constitutive of something being a certain connective. If you think, e.g., that CO and CT, but not CM, are constitutive of consequence and that FU isn't constitutive of conjunction but PF and PEM are constitutive of disjunction, then *Rej(Mixed-Cut,DDT,CM,FU)* is probably a nonmonotonic logic that you should consider.

If you are a logical inferentialist, you think that something has the meaning of, say, a conditional just in case, and because, it has a particular inferential role. In that case, you may plausibly think that principles like DDT, PF, and FU are constitutive of our connectives meaning what we want them to mean. For these are principles that plausibly characterize a part of the inferential roles of the conditional, disjunction and conjunction, respectively. Such an inferentialist view seems attractive to me and, hence, I will focus on options that are attractive from an inferentialist perspective. Before we get to this, however, I want to point out another issue for which inferentialist inclinations are relevant for the choice of a nonmonotonic logic, namely what we think logic should do for us.

## 4 Sticking to logic

If you are an inferentialist, it is probably not only the extension of your nonmonotonic consequence relation that will matter to you but also the way it is generated. You will probably say that the inferential roles of logically complex sentences are determined by the inferential rules governing the logical connectives, together with the inferential roles of the atomic sentences. Hence, you will want your nonmonotonic consequence relation to be generated by putting together a consequence relation over atomic sentences and rules governing the inferential behavior of the logical connectives. In this section, I will discuss implications of this view for the choice of a nonmonotonic logic.

We want to codify inference like that from "Zazzles is a cat" to "Zazzles has four legs." Hence, our consequence relation cannot be formal, in the sense of being closed under substitution (see Makinson, 2003). This isn't surprising. The inference about Zazzles has something to do with cats and legs and the meanings of “cat” and “legs.” Substantive information about cats and legs and the meanings of “cat” and “legs” is curled up in our inference about Zazzles.

The logician has no particular expertise regarding cats and legs or the meanings of “cat” and “legs.” So in order to construct a consequence relation in which “Zazzles has four legs” (or a sentence that stands for it in an artificial language) is a consequence of “Zazzles is a cat” (or a sentence that stands for it in an artificial language), the logician must draw on extra-logical knowledge. In particular, the atomic fragment of our consequence relation must be given to the logician from elsewhere. The logician gets no say in determining it. After all, implication relations among atomic sentences always embody substantive connections regarding which the logician cannot claim any expertise.

Of course, the logician may only be interested in atomic consequence relations that have certain structural properties, like CO, RE, CT or CM. But in order to justify such a restrictions from a philosophical perspective, we must give a philosophical interpretation of consequence, i.e. of the turnstile, and we must motivate the idea that all atomic consequence relations have certain properties on the basis of that interpretation.

This point is important because many nonmonotonic logicians are motivated by problems in artificial intelligence. For the purposes of artificial intelligence, however, it doesn’t matter when and how we bring in extra-logical information, as long as we solve the practical problems at hand. We may, e.g., want a machine to deduce as much useful and reliable information as possible, in whatever way possible.

So in choosing a nonmonotonic logic, you should ask yourself: Do you allow your logic to partly determine the atomic fragment of your consequence relation? Or do you want a logic that conservatively extends any consequence relation over atomic sentences? If you want a maximally powerful inference engine, the first choice is preferable. If, however, you want your logic to determine the inferential roles of logically complex sentences in terms of the inferential roles of simpler sentences, then the second choice is preferable.

Comparing default logic and KLM can help to clarify my point. In default logic, we encode extra-logical information into default rules, which allow us to add a conclusion to our current knowledge base if we can get the so-called prerequisite and all the so-called justifications are consistent with our current knowledge base. We then apply our default rules until we reach a fixed point.[8] Suppose, e.g., we know that Zazzles is a cat and that Zazzles likes to sleep on the sofa. Our only default rule is one that allows us to infer that something is hairy if it is a cat and it is consistent with our knowledge base that Zazzles is hairy. According to default logic, we can infer that Zazzles is hairy. And we can do that without ever explicitly feeding our logic the information that {"Zazzles is a cat", "Zazzles likes to sleep on the sofa"} nonmonotonically implies "Zazzles is hairy." Thus, default logic partly determines the consequence relation over atomic sentences.

Preferential semantics, in the spirit of KLM, differs in this respect from default logic. In KLM, we start with a partial order over states. We say that $\Gamma \mid\!\sim A$ iff all the minimal states in which all members of $\Gamma$ are true are also states in which $A$ is true. We can think of the partial order as representing the extra-logical information which we are feeding into our cumulative logic. This ordering can be given antecedently to the semantic clauses that embody an account of the logical constants. If we interpret KLM in this way, we say, in effect, that all the extra-logical information must be given before we determine the consequence relation over logically complex sentences.[9] Where KLM diverges from inferentialism is in giving a model-theoretic account of the meanings of the logical constants.

The question whether you want a nonmonotonic logic that is suitable for purposes in artificial intelligence or one that is compatible with inferentialism isn't a question about the extension of your consequence relation. Rather, it is a question about what you take to be basic. If you are an inferentialist, you should take nonmonotonic consequence relations over atomic sentences to be basic. Given that nonmonotonic logics that are suitable for artificial intelligence are well known, I want to close by presenting two options that are attractive from an inferentialist perspective—both in the extension of their consequence relations and in how they are generated.

## 5 Two inferentialist nonmonotonic logics

In this section, I will present two nonmonotonic logics that serve as examples of what I consider inferentialism-friendly nonmonotonic logics. I have

[8] That procedure defines an extension of our knowledge base. Skeptical consequences of our knowledge base are then those sentences that are in every extension. And credulous consequences are those that are in at least one extension.

[9] That is why, in KLM, "Nixon is a pacifist" does not follow from "Nixon is a Quaker and Nixon is a Republican", unless we order the states in such a way that in all the minimal states in which Nixon is a Quaker and a Republican, he is also a pacifist (Kraus et al., 1990).

presented a similar logic elsewhere (Hlobil, 2016); so I will focus on what is new and skip details that can be found in previous presentations of the framework.

### 5.1 The basics

Let $\mathfrak{L}_0$ be a set of atomic sentences, and let $\mathfrak{L}$ be the result of adding the connectives $\rightarrow, \vee$, and $\&$ (and $\neg$ in the second logic; it can be treated as defined in the first logic) to this atomic language, with their usual syntax.

Let's start with a (multiple conclusion) nonmonotonic consequence relation over atoms.[10]

**Definition 1** *A base consequence relation,* $|\!\sim_0 \subseteq \mathcal{P}(\mathfrak{L}_0) \times \mathcal{P}(\mathfrak{L}_0)$*, is a relation between sets of atomic sentences.*

This definition is maximally liberal. In the context of a particular logic, we will add constraints. If you are an inferentialist, you will think that such base consequence relations define (at least in part) the meanings of the atomic sentences. We will use our base consequence relation as a set of axioms in otherwise familiar sequent calculi. The resulting consequence relation over the whole language is $|\!\sim$.

### 5.2 Making G3cp nonmonotonic

Suppose we want a supra-classical nonmonotonic logic. In that case, we should choose sequent rules for classical logic. However, we don't want to enforce structural principles like monotonicity. Hence, we should choose a sequent calculus for classical logic in which Weakening is absorbed.

The sequent calculus G3cp fits the bill (Troelstra & Schwichtenberg, 2000). The axioms of G3cp are $\Gamma, p \vdash p, \Delta$ and $\bot, \Gamma \vdash \Delta$. Like in LK, it suffices in G3cp to use just the atomic instances of these axioms to get classical logic. So if we make sure that all of our base relations include all atomic instances of the axioms of G3cp, our resulting nonmonotonic logic will be supra-classical. Let's say that base relations that include all the axioms of G3cp are "NM-G3cp-fit."

**Definition 2** *A* NM-G3cp-fit base consequence relation *is a base consequence relation that includes all the atomic instances of* $\Gamma, p|\!\sim_0 p, \Delta$ *and* $\bot, \Gamma|\!\sim_0 \Delta$ *(and possibly more sequents).*

[10] This is similar to what is done in work on atomic systems (Piecha & Schroeder-Heister, 2016; Sandqvist, 2015). In this literature, it is common to allow higher-order rules and to enforce so-called "definitional reflection." I do neither of these two things here.

**Axioms of NM-G3cp**

If $\Gamma \mid\!\sim_0 \Delta$, then $\Gamma \mid\!\sim \Delta$ is an axiom.

**Rules of NM-G3cp (which are identical to those of G3cp)**

$$\frac{\Gamma \mid\!\sim A, \Delta \qquad \Gamma, B \mid\!\sim \Delta}{\Gamma, A \to B \mid\!\sim \Delta}\ \text{LC} \qquad \frac{\Gamma, A \mid\!\sim B, \Delta}{\Gamma \mid\!\sim A \to B, \Delta}\ \text{RC}$$

$$\frac{\Gamma, A, B \mid\!\sim \Delta}{\Gamma, A\&B \mid\!\sim \Delta}\ \text{L\&} \qquad \frac{\Gamma \mid\!\sim A, \Delta \qquad \Gamma \mid\!\sim B, \Delta}{\Gamma \mid\!\sim A\&B, \Delta}\ \text{R\&}$$

$$\frac{\Gamma, A \mid\!\sim \Delta \qquad \Gamma, B \mid\!\sim \Delta}{\Gamma, A \vee B \mid\!\sim \Delta}\ \text{Lv} \qquad \frac{\Gamma \mid\!\sim A, B, \Delta}{\Gamma \mid\!\sim A \vee B, \Delta}\ \text{Rv}$$

Figure 2: NM-G3cp

If we want to construct a particular nonmonotonic logic, we have to choose a particular NM-G3cp-fit base relation. We should choose the base relation that determines, to the extent that consequence relations do that, the meanings for the atomic sentences that we want them to have. We can now define the nonmonotonic logic NM-G3cp.

**Definition 3** NM-G3cp: *A logic belongs to the family NM-G3cp just in case its consequence relation, $\mid\!\sim$, is the smallest set of sequents that closes an NM-G3cp-fit base consequence relation under the rules of NM-G3cp.*

Figure 2 gives a sequent calculus for NM-G3cp. NM-G3cp has a couple of nice properties. First, it is supra-classical. Second, NM-G3cp extends NM-G3cp-fit base relations conservatively. To see this, notice that all the rules of NM-G3cp conclude sequents that contain logical constants. Third, like in G3cp, all the rules are invertible.

**Proposition 1** *All the rules of NM-G3cp are invertible; i.e., if the root sequent of a rule application holds, then so do the top sequents.*

*Proof.* By induction on proof-height. I will do just one subcase of L&, namely the one where the root comes by LC. Suppose that $\Gamma, A\&B, C \to D \mid\!\sim \Delta$ comes by LC. The premises are $\Gamma, A\&B, \mid\!\sim C, \Delta$ and $\Gamma, A\&B, D \mid\!\sim \Delta$. By our induction hypothesis, this means that we can derive $\Gamma, A, B, \mid\!\sim C, \Delta$ and $\Gamma, A, B, D \mid\!\sim \Delta$. By LC, we get $\Gamma, A, B, C \to D \mid\!\sim \Delta$. The other cases are analogous. □

The invertibility of the rules implies that if we have an effective procedure for checking whether a set of atomic sequents is in our base relation, then the NM-G3cp extension allows for effective root-first proof search.

A further attractive consequence of the invertibility of the G3cp rules is that the connectives satisfy DDT, PF, and FU. That is good news for inferentialists who think that DDT, PF and FU are constitutive of the meanings of the conditional, disjunction, and conjunction respectively.

NM-G3cp is of the type *Rej(Mixed-Cut,CT,CM,DI)*. After all, it is supraclassical (so CO and PEM hold) and it obeys DDT, FU, and PF. Hence, Cautious Monotonicity, Cumulative Transitivity and Distribution all fail in NM-G3cp.

### 5.3 Making LR nonmonotonic

We have seen in the previous subsection that NM-G3cp obeys neither Mixed-Cut nor CT. That will certainly raise some eyebrows. Although nontransitive logics are becoming more common (Cobreros, Egré, Ripley, & van Rooij, 2012; Ripley, 2013), some people will insist on Cut. In this section, I show how we can construct a nonmonotonic logic whose proof-theory is as elegant as that of NM-G3cp while it also satisfies DDT and Cut. In order to do so, we must turn to relevance logic, as is obvious from Figure 1 because we must reject CO in order to jointly satisfy DDT and CT.

It is well known that the most prominent relevance logics E and R don't have simple sequent calculus formulations because they obey distribution. The distribution-free relevance logic LR, however, has an elegant sequent calculus formulation (Bimbó, 2015). So let's follow the strategy we used to formulate NM-G3cp in order to turn LR into the nonmonotonic logic NM-LR.

The axioms of LR are all the instances of RE (i.e. $A \mid\!\sim A$). For our logic to be as strong as LR, we must ensure that our logic proves all these instances. We can do so by, first, requiring that all base relations include all atomic instances of RE and, second, ensuring that our rules allow us to

**Axioms of NM-LR**

If $\Gamma \mid\!\sim_0 \Delta$, then $\Gamma \mid\!\sim \Delta$ is an axiom.

**Rules of NM-LR (which are identical to those of LR)**

$$\frac{\Gamma \mid\!\sim A, \Delta \qquad \Theta, B \mid\!\sim \Lambda}{\Gamma, \Theta, A \to B \mid\!\sim \Delta, \Lambda}\ \text{LC} \qquad \frac{\Gamma, A \mid\!\sim B, \Delta}{\Gamma \mid\!\sim A \to B, \Delta}\ \text{RC}$$

$$\frac{\Gamma \mid\!\sim A, \Delta}{\Gamma, \neg A \mid\!\sim \Delta}\ \text{LN} \qquad \frac{\Gamma, A \mid\!\sim \Delta}{\Gamma \mid\!\sim \neg A, \Delta}\ \text{RN}$$

$$\frac{\Gamma, A_l \mid\!\sim \Delta}{\Gamma, A_1 \& A_2 \mid\!\sim \Delta}\ \text{L\&}\ (l{=}1 \text{ or } 2) \qquad \frac{\Gamma \mid\!\sim A, \Delta \qquad \Gamma \mid\!\sim B, \Delta}{\Gamma \mid\!\sim A \& B, \Delta}\ \text{R\&}$$

$$\frac{\Gamma, A \mid\!\sim \Delta \qquad \Gamma, B \mid\!\sim \Delta}{\Gamma, A \vee B \mid\!\sim \Delta}\ \text{Lv} \qquad \frac{\Gamma \mid\!\sim A_l, \Delta}{\Gamma \mid\!\sim A_1 \vee A_2, \Delta}\ \text{Rv}\ (l{=}1 \text{ or } 2)$$

Figure 3: NM-LR

derive all the logically complex instances of RE. With a view to the first point, we define NM-LR-fit base relations.

**Definition 4** *A* NM-LR-fit base consequence relation *is a base consequence relation*, $\mid\!\sim_0$, *that includes all atomic instances of RE (i.e.* $p \mid\!\sim_0 p$*).*

We can now simply use the sequent rules of LR (with the exception that we ignore the rules for contraction because we are working with sets). After all, the rules of LR preserve Reflexivity. The resulting system, NM-LR, is set out in Figure 3.

NM-LR includes LR because we get all the axioms of LR (which can easily be shown by induction on complexity). By inspecting the rules, we can see that NM-LR defines a conservative extension. For, as in NM-G3cp, each rule concludes a sequent with a logical connective. As desired, Cut is admissible in NM-LR if the base relation we choose satisfies Cut.

**Proposition 2** *Mixed-Cut is admissible in any NM-LR extension of a NM-LR-fit base relation that satisfies Mixed-Cut.*

*Proof.* The known Cut-elimination proof for LR, which works basically like Gentzen's original proof for LK (Bimbó, 2015), shows that Mixed-Cut can be pushed up in proof-trees of NM-LR. Hence, it suffices to note that the axioms are closed under Mixed-Cut. □

Notice that NM-LR gives us relevance logics that are ampliative (if the base relation is ampliative). As intimated in Section 2, nonmonotonic logics in the tradition of default logic and KLM want to have more consequences than classical logic, whereas relevance logicians want to have fewer consequences than classical logic. The consequence relations of NM-LR can do both of these things at the same time. We can add risky inferences, and we can do this without getting, e.g., paradoxes of material implication.

## 6 Conclusion

I have described an exhaustive menu of options from which you can choose a nonmonotonic logic. All nonmonotonic logics fall into at least one of the types charted in Figure 1. Along the way, I have presented a sequent calculus approach to nonmonotonic logic, and I have given two examples of logics that this approach yields, NM-G3cp and NM-LR. I have argued that these logics have some features that are attractive for inferentialists.

## References

Anderson, A. R., & Belnap, N. D. (1975). *Entailment: The Logic of Relevance and Necessity*. Princeton: Princeton University Press.

Arieli, O., & Avron, A. (2000). General patterns for nonmonotonic reasoning: from basic entailments to plausible relations. *Logic Journal of the IGPL*, *8*(2), 119–148.

Batens, D. (2007). A universal logic approach to adaptive logics. *Logica Universalis*, *1*(1), 221–242.

Bimbó, K. (2015). *Proof theory: Sequent calculi and related formalisms*. Boca Raton: CRC Press.

Cobreros, P., Egré, P., Ripley, D., & van Rooij, R. (2012). Tolerant, classical, strict. *Journal of Philosophical Logic*, *41*(2), 347–385.

Dung, P. M. (1995). On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games. *Artificial Intelligence*, *77*, 321–358.

Dunn, M., & Restall, G. (2002). Relevance logic. In D. Gabbay & F. Guenthner (Eds.), *Handbook of Philosophical Logic* (Vol. 6, pp. 1–128). Kluwer Academic Publishers.

Hlobil, U. (2016). A nonmonotonic sequent calculus for inferentialist expressivists. In P. Arazim & M. Dančák (Eds.), *The Logica Yearbook 2015* (pp. 87–105). College Publications.

Kraus, S., Lehmann, D., & Magidor, M. (1990). Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence*, *44*, 167-207.

Makinson, D. (2003). Bridges between classical and nonmonotonic logic. *Logic Journal of the IGPL*, *11*(1), 69–96.

Piecha, T., & Schroeder-Heister, P. (2016). Atomic systems in proof-theoretic semantics: Two approaches. In J. Redmond, O. Pombo Martins, & Á. Nepomuceno Fernández (Eds.), *Epistemology, Knowledge and the Impact of Interaction* (pp. 47–62). Cham: Springer.

Reiter, R. (1980). A logic for default reasoning. *Artificial Intelligence*, *13*(1–2), 81–132. (Special Issue on Non-Monotonic Logic)

Ripley, D. (2013). Paradoxes and failures of Cut. *Australasian Journal of Philosophy*, *91*(1), 139–164.

Sandqvist, T. (2015). Base-extension semantics for intuitionistic sentential logic. *Logic Journal of the IGPL*, *23*(5), 719–731.

Strasser, C., & Antonelli, G. A. (2016). *Non-monotonic Logic.* The Stanford Encyclopedia of Philosophy (Winter 2016 Edition), Edward N. Zalta (ed.),. Retrieved from `https://plato.stanford.edu/ archives/ win2016/ entries/ logic-nonmonotonic/`

Troelstra, A. S., & Schwichtenberg, H. (2000). *Basic proof theory* (2nd ed.). Cambridge University Press.

Ulf Hlobil
Concordia University
Canada
E-mail: `ulf.hlobil@concordia.ca`